TMA4170 Fourier Analysis

Fourier transform: $\mathcal{F}[f](\xi) := \begin{cases} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx & \text{for } f \in L^1(\mathbb{R}) \\ \underset{f_n \xrightarrow[L^2]{} f}{L^2\text{-}\lim} \hat{f}_n(\xi) & \text{for } f \in L^2(\mathbb{R}) \end{cases}$

$$\boxed{\overline{\mathcal{F}}[g](x) := \overline{\mathcal{F}[g](x)} = \int_{\mathbb{R}} g(\xi)\, e^{2\pi i x \xi}\, d\xi = \mathcal{F}[g(-\xi)](x) = \mathcal{F}[g](-x) \ \left(= \mathcal{F}^{-1}[g](x)\right)}$$

A. In $S(\mathbb{R})$: $\mathcal{F} : S(\mathbb{R}) \longrightarrow S(\mathbb{R})$ invertible isometry w.r.t. $\|\cdot\|_2$

$f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R})$, $\quad f(x) = \underset{\overline{\mathcal{F}}}{\overset{\shortparallel}{\mathcal{F}^{-1}}}[\hat{f}](x)$, $\quad \|f\|_2 = \|\hat{f}\|_2$, $\quad \widehat{f * g} = \hat{f} \cdot \hat{g}$

B. In $L^2(\mathbb{R})$: $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ invertible isometry w.r.t. $\|\cdot\|_2$

$f \in L^2(\mathbb{R}) \Rightarrow \hat{f} \in L^2(\mathbb{R})$, $\quad f(x) \overset{L^2}{=} \underset{\overline{\mathcal{F}}}{\overset{\shortparallel}{\mathcal{F}^{-1}}}[\hat{f}](x)$, $\quad \|f\|_2 = \|\hat{f}\|_2$, $\quad \widehat{f * g} = \hat{f} \cdot \hat{g}$

C. In $L^1(\mathbb{R})$: $\mathcal{F} : L^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$, not invertible everywhere/isometry

or $C_m(\mathbb{R})$

$f \in L^1 \implies \hat{f} \in C_0$, but $\notin L^1$, $\quad |\hat{f}(\xi)| \xrightarrow[|\xi|\to\infty]{} 0$, $\quad \|\hat{f}\|_\infty \leq \|f\|_1$, $\quad \cancel{\|f\|_2 = \|\hat{f}\|_2}$

Inversion:

① (Global) when $\hat{f} \in L^1$: $\quad f, \hat{f} \in L^1 \implies f(x) = \underset{\overline{\overline{\mathcal{F}}}}{\mathcal{F}^{-1}}[\hat{f}](x)$

② "Fejer" type when $\hat{f} \notin L^1$: $f \in L^1 \implies f(x) = \lim_{\delta\to 0} \underset{\overline{\overline{\mathcal{F}}}}{\mathcal{F}^{-1}}[\hat{K}_\delta \cdot \hat{f}](x)$, $\quad \hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$

③ Pointwise where f smooth:

$f \in PC_m(L^1)$; $f(x^\pm)$, $D^\pm f(x)$ exists $\implies \underset{\overline{\overline{\mathcal{F}}}}{\mathcal{F}^{-1}}[\hat{f}](x) = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$

Properties of Fourier transform

(a) $\mathcal{F}[f(x-h)](\xi) = e^{-2\pi i \xi h}\hat{f}(\xi)$ and $\mathcal{F}[e^{2\pi i x h} f(x)](\xi) = \hat{f}(\xi - h)$

(b) $\mathcal{F}[f(\delta x)](\xi) = \delta^{-1}\hat{f}(\delta^{-1} x)$

(c) $\mathcal{F}[f'](\xi) = 2\pi i \xi \hat{f}(\xi)$ and $\mathcal{F}[-2\pi i x f(x)](\xi) = \hat{f}'(\xi)$

(d) $\mathcal{F}[f * g](\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$ and $\mathcal{F}[f \cdot g](\xi) = \hat{f} * \hat{g}(\xi)$

(e) $\mathcal{F}^{-1}[g](x) = \mathcal{F}[g(-\xi)](x) = \mathcal{F}[g](-x) = \overline{\mathcal{F}}[g](x)$

$\boxed{\mathcal{F}^{-1} \text{ satisfy the same properties}}$

f	$K(x) = e^{-\pi x^2}$	$e^{-2\pi\lvert x\rvert}$	$\frac{1}{1+x^2}$	$\frac{\sin x}{x}$	$\chi_{[-1,1]}(x)$
$\hat{f}$	$\hat{K}(\xi) = K(\xi)$	$\frac{1}{\pi} \cdot \frac{1}{1+\xi^2}$	$\pi e^{-2\pi\lvert\xi\rvert}$	$\pi \chi_{[-1,1]}(\xi)$	$\frac{1}{\pi} \cdot \frac{\sin \xi}{\xi}$
$\check{f}$	$\check{K}(y) = K(y)$	$\frac{1}{\pi} \cdot \frac{1}{1+y^2}$	$\pi e^{-2\pi\lvert y\rvert}$	$\pi \chi_{[-1,1]}(y)$	$\frac{1}{\pi} \cdot \frac{\sin y}{y}$

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Poisson summation formula:

$$f(x) \in S(\mathbb{R}) \implies \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}, \quad \hat{f}(n) = \mathcal{F}[f](n)$$

$$\overset{x=0}{\implies} \boxed{\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)}$$

Periodization of $f(x)$: $\quad F_1(x) := \sum_{n\in\mathbb{Z}} f(x+n), \quad F_2(x) = \sum \hat{f}(n)e^{2\pi i n x}$

Obs: F_1, F_2 1-periodic, continuous; $\hat{F}_1(m) = \hat{f}(m) = \hat{F}_2(m) \ \forall m \in \mathbb{Z}$

Poisson $\iff F_1(x) = F_2(x) \ \forall x \in \mathbb{R}$